FRACTAL ANALYSIS FOR SETS OF NON-DIFFERENTIABILITY OF MINKOWSKI'S QUESTION MARK FUNCTION

MARC KESSEBÖHMER AND BERND O. STRATMANN

ABSTRACT. In this paper we study various fractal geometric aspects of the Minkowski question mark function Q. We show that the unit interval can be written as the union of the three sets $\Lambda_0 := \{x : Q'(x) = 0\}$, $\Lambda_\infty := \{x : Q'(x) = \infty\}$, and $\Lambda_\sim := \{x : Q'(x)$ does not exist and $Q'(x) \neq \infty\}$. The main result is that the Hausdorff dimensions of these sets are related in the following way.

$$\dim_H(\nu_F) < \dim_H(\Lambda_\sim) = \dim_H(\Lambda_\infty) = \dim_H\left(\mathscr{L}(h_{\mathrm{top}})\right) < \dim_H(\Lambda_0) = 1.$$

Here, $\mathscr{L}(h_{\mathrm{top}})$ refers to the level set of the Stern-Brocot multifractal decomposition at the topological entropy $h_{\mathrm{top}} = \log 2$ of the Farey map F, and $\dim_H(\nu_F)$ denotes the Hausdorff dimension of the measure of maximal entropy of the dynamical system associated with F. The proofs rely partially on the multifractal formalism for Stern-Brocot intervals and give non-trivial applications of this formalism.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we return to the origins of the multifractal analysis of measures, which started with work on fractal sets by Mandelbrot and others in the 1980s (see e.g. [12], [13], [23], [9]). For this, we go even further back in time, and consider a function Q of the unit interval $\mathscr{U}$ into itself, which was originally designed by Minkowski [25] in order to illustrate the Lagrange property of quadratic surds. Today, this function is usually referred to as the Minkowski question mark function, and it appears in various different disguises. For instance, it appears as the distribution function of the measure of maximal entropy ν_F for the dynamical system arising from the Farey map F. That is,

$$Q(x) = \nu_F([0,x)), \text{ for all } x \in \mathscr{U}.$$

Since the support of ν_F is equal to $\mathscr{U}$, and since ν_F is singular with respect to the 1-dimensional Lebesgue measure λ on $\mathscr{U}$ (see Salem [30]), the graph of Q is appropriately described by the term 'slippery devil's staircase', a term which was coined by Gutzwiller and Mandelbrot in [12] (see also [11], [1]). Another disguise of Q is, that it provides a stable bridge between the Farey system and the binary system $(\mathscr{U}, T)$, that is the dynamical system which arises from the tent map T. In this disguise, the homeomorphism Q represents the topological conjugacy map between the Farey system and the tent system, such that $T \circ Q = Q \circ F$. Using elementary observations for the regular continued fraction

Date: 1st February 2008.
2000 *Mathematics Subject Classification.* 26A30, 10K50.
Key words and phrases. Minkowski question mark function, singular functions, Stern-Brocot spectrum, Farey map.

expansion $x=[a_1,a_2,\ldots]$ of elements $x\in\mathscr{U}$, one readily rediscovers the following alternating sum representation of Q, first obtained by Denjoy [5] (see also [6] , [30], [28], [29]),

$$Q(x):=-2\sum_{k\in\mathbb{N}}(-1)^k\,2^{-\sum_{i=1}^k a_i},\text{ for all } x=[a_1,a_2,\ldots]\in\mathscr{U}.$$

These observations mark the starting point for the fractal geometric analysis of the function Q in this paper. We will show that interesting measure theoretical aspects of the Minkowski scenario can be derived from the recently obtained multifractal analysis for Stern-Brocot intervals [19]. As a first demonstration of the fruitfulness of this approach, we study fractal geometric relationships between Q, ν_F and the Gauss measure m_G. We obtain the result that one can explicitly compute the integral over Q with respect to m_G, as well as the integral with respect to ν_F over the distribution function Δ_{m_G} of m_G. That is, with $\dim_H$ referring to the Hausdorff dimension, we obtain

$$\int_{\mathscr{U}} Q\,dm_G = 1-\int_{\mathscr{U}}\Delta_{m_G}\,d\nu_F = (\dim_H(\nu_F)-1/2)/\dim_H(\nu_F)\ (\approx 0.571612).$$

As an immediate consequence of this, one can then also rediscover a result by Kinney [21] which expresses the Hausdorff dimension of ν_F in terms of a certain explicit integral.

Subsequently, we draw the attention to the derivative Q' of Q. It was shown only relatively recently in [26] that if $Q'(x)$ exists in the generalised sense, meaning that $Q'(x)$ either exists or is equal to infinity, then $Q'(x)$ either vanishes or else is equal to infinity. We give a new and very elementary proof of this fact, and then add to this by showing that $Q'(x)$ is equal to infinity if and only if $\lim_{n\to\infty}\nu_F(T_n(x))/\lambda(T_n(x))$ is equal to infinity. Here, $T_n(x)$ refers to the unique atom of the n-th refinement of $\mathscr{U}$ with respect to F, such that $x\in T_n(x)$. Moreover, we show that if for the approximants p_k/q_k of $x=[a_1,a_2,\ldots]$ we have $\lim_{k\to\infty}a_{k+1}\cdot\nu_F\left([p_k/q_k,p_{k+1}/q_{k+1})_\pm\right)/\lambda\left([p_k/q_k,p_{k+1}/q_{k+1})_\pm\right)=0$, then $Q'(x)$ vanishes (see Section 5 for the definition of $[\ ,\ )_\pm$). The latter, slightly technical observations will turn out to be crucial in the multifractal analysis of Q' to come. In order to state the main results of this analysis, note that $\mathscr{U}$ can be decomposed into mutually disjoint sets as follows.

$$\mathscr{U}=\Lambda_0\cup\Lambda_\infty\cup\Lambda_\sim,$$

where $\Lambda_0:=\{x:Q'(x)=0\}$, $\Lambda_\infty:=\{x:Q'(x)=\infty\}$, and $\Lambda_\sim$ refers to the set of elements for which Q' does not exist in the generalised sense. Surprisingly, before these investigations relatively little was known about this decomposition. The main contributions thus far were made by Salem, and these date back more than 60 years. In our notation, the aforementioned result of Salem [30] reads as $\lambda(\Lambda_0)=1$. More precisely, Salem [30] showed that if $Q'([a_1,a_2,\ldots])$ exists and is equal to some finite value, and if, additionally, $\limsup_{n\to\infty}a_n=\infty$, then $[a_1,a_2,\ldots]\in\Lambda_0$. The analysis in this paper will give significant extensions of this classical result. In order to state these extensions, recall that in [19] we computed the dimension spectrum of the multifractal decomposition

$$\mathscr{L}(s):=\left\{x\in\mathscr{U}:\lim_{n\to\infty}\frac{\log\lambda(T_n(x))}{\log\nu_F(T_n(x))}=\frac{s}{h_{\mathrm{top}}}\right\}.$$

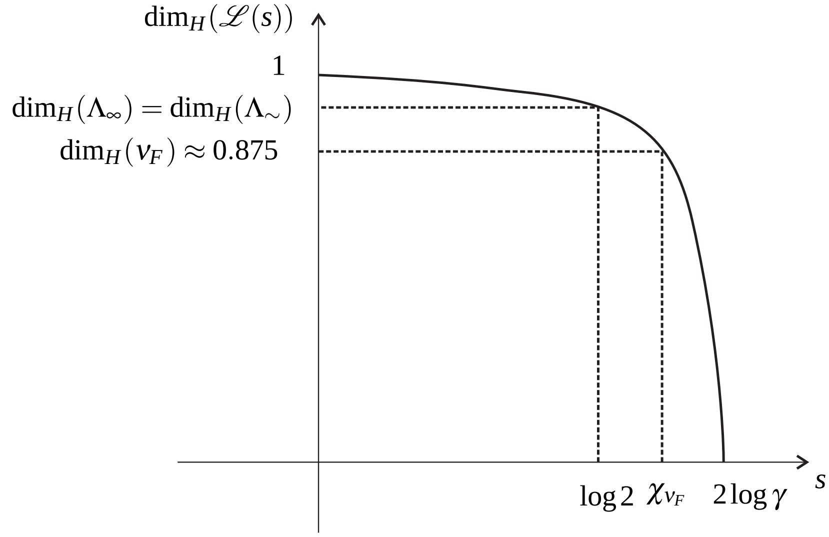

FIGURE 1.1. The Stern-Brocot dimension spectrum

Here, $h_{\text{top}} = \log 2$ refers to the topological entropy of the Farey map F. In particular, in [19] it was shown that the Hausdorff dimension of $\mathscr{L}(s)$ is nontrivial if and only if $s \in [0, 2\log\gamma)$ (with γ referring to the Golden Mean). By relating this multifractal decomposition to the Minkowski scenario in this paper, a first outcome is that

$$\mathscr{L}(s) \subset \Lambda_\infty \text{ for } s \in (h_{\text{top}}, 2\log\gamma], \text{ whereas } \mathscr{L}(s) \subset \Lambda_0 \text{ for } s \in [0, h_{\text{top}}).$$

By expressing this result in terms of the convergents p_k/q_k of elements $x = [a_1, a_2, \ldots]$, one then immediately derives the following result.

$$\left\{ x : \lim_{n\to\infty} 2\log q_n / \sum_{i=1}^n a_i > h_{\text{top}} \right\} \subset \Lambda_\infty, \text{ and } \left\{ x : \lim_{n\to\infty} 2\log q_n / \sum_{i=1}^n a_i < h_{\text{top}} \right\} \subset \Lambda_0.$$

Let us now finally come to the main result of this paper. For this, note that on the basis of the results of Denjoy and Salem, one might suspect that the complement of Λ_0 in $\mathscr{U}$ can still be large, in the sense that its Hausdorff dimension could be equal to one. Our main result now shows that this is in fact not the case. More precisely, for the Hausdorff dimensions of Λ_∞ and $\Lambda_\sim$, we obtain the result

$$0.875 \approx \dim_H(\nu_F) < \dim_H(\Lambda_\sim) = \dim_H(\Lambda_\infty) = \dim_H\left(\mathscr{L}(h_{\text{top}})\right) < \dim_H(\Lambda_0) = 1.$$

Here, the proof of the second equality $\dim_H(\Lambda_\infty) = \dim_H\left(\mathscr{L}(h_{\text{top}})\right)$ is derived from a non-trivial application of the multifractal formalism for Stern-Brocot intervals obtained in [19] (cf. Figure 1.1), whereas the proof of the first equality $\dim_H(\Lambda_\sim) = \dim_H(\Lambda_\infty)$ combines this formalism with an extension of the analysis of sets of 'non-typical' points in [2] to non-hyperbolic dynamical systems.

Remark 1.1. In contrast to 'ordinary devil's staircases', which usually arise from distribution functions of fractal measures on Cantor-like sets, a *slippery devil's staircase* is the graph of the distribution function of a measure whose support is equal to the whole unit interval $\mathscr{U}$, but which is nevertheless

singular with respect to the Lebesgue measure λ on $\mathscr{U}$. Slippery devil's staircases should not be confused with ordinary devil's staircases. In order to give a brief demonstration of the difference between these two types of staircases, let us consider the example of the homogeneous Cantor measure $\mu_{\mathscr{C}}$ supported on Cantor's ternary set $\mathscr{C}$. It is immediately clear that the derivative of the distribution function $\Delta_{\mu_{\mathscr{C}}}$ vanishes on the complement of $\mathscr{C}$ in $\mathscr{U}$, giving that $\lambda(\Lambda_0(\Delta_{\mu_{\mathscr{C}}})) = 1$. By a result of Darst [4] (see also [8]), one has $\dim_H(\Lambda_\sim(\Delta_{\mu_{\mathscr{C}}})) = (\dim_H(\mathscr{C}))^2$. Moreover, by a classical result of Gilman [10] we have that if the derivative of $\Delta_{\mu_{\mathscr{C}}}$ exists in the generalised sense at some point $x \in \mathscr{C}$, then it can only be equal to infinity. Hence, $\dim_H(\Lambda_\infty(\Delta_{\mu_{\mathscr{C}}})) = \dim_H(\mathscr{C})$. Let us remark that the result of Darst can be derived from straightforward adaptations of techniques developed for estimating the Hausdorff dimension of well-approximable irrational numbers (see e.g. [16], [32]). Hence, in this situation, the set $\Lambda_\sim$ can be thought of as being conceptionally analogous to the set of well-approximable numbers. This analogy no longer holds for slippery devil's staircases.

2. Multifractal formalism for Stern-Brocot intervals revisited

Let us first recall the classical construction of Stern-Brocot intervals in the unit interval $\mathscr{U} := [0,1]$ ([31], [3], see also [14], [15], [27]). For each $n \in \mathbb{N}_0$, the elements of the Stern-Brocot sequence $\{s_{n,k}/t_{n,k} : k = 0,\dots,2^n\}$ of order n are defined recursively for $n \in \mathbb{N}$, $k = 0,\dots,2^{n-1}$ and $r = s,t$ as follows

$$s_{0,0} := 0, s_{0,1} := t_{0,0} := t_{0,1} := 1, r_{n,2k} := r_{n-1,k} \text{ and } r_{n,2k-1} := r_{n-1,k-1} + r_{n-1,k}.$$

With this ordering of the rationals in $\mathscr{U}$ we define the set $\mathscr{T}_n$ of Stern-Brocot intervals of order n by

$$\mathscr{T}_n := \{T_{n,k} := [s_{n,k}/t_{n,k}, s_{n,k+1}/t_{n,k+1}) : k = 0,\dots,2^n - 1\}.$$

Clearly, $\mathscr{T}_n$ is the set of atoms of the n-th refinement of $\mathscr{U}$ with respect to the Farey map, and one immediately finds that for each $x \in \mathscr{U}$ and $n \in \mathbb{N}_0$ there exists a unique Stern-Brocot interval $T_n(x) \in \mathscr{T}_n$ such that $x \in T_n(x)$.

In [19] (see also [17] [18]), we considered the n-th Stern-Brocot quotient ℓ_n and the Stern-Brocot growth rate ℓ, which are given by (assuming that the limit exists)

$$\ell_n(x) := \frac{1}{n} \log\left(1/\lambda\left(T_n(x)\right)\right) \text{ and } \ell(x) := \lim_{n\to\infty} \ell_n(x).$$

Here, λ refers to the 1–dimensional Lebesgue measure on $\mathscr{U}$.

One of the main results in [19] determined the Lyapunov spectrum arising from ℓ. That is, we computed the Hausdorff $\dim_H$ of the following level sets

$$\mathscr{L}(s) := \{x \in \mathscr{U} : \ell(x) = s\}, \text{ for } s \in \mathbb{R}.$$

For the purposes of this paper the following main results of [19] will be crucial. Here, P refers to the Stern-Brocot pressure function P, which is given for $t \in \mathbb{R}$ by

$$P(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{T \in \mathscr{T}_n} (\lambda(T))^t, \tag{2.1}$$

and $\widehat{P}$ refers to the Legendre transform, given for $s \in \mathbb{R}$ by $\widehat{P}(s) := \sup_{t\in\mathbb{R}}\{st - P(t)\}$. Also, throughout we let $\gamma := (\sqrt{5}+1)/2$, and use the convention $\widehat{P}(0)/0 := -1$.

Theorem 2.1 ([19]). *For each* $s \in [0, 2\log\gamma]$, *we have*

$$\dim_H(\mathscr{L}(s)) = -\frac{\widehat{P}(-s)}{s} \; (=: d(s)).$$

Here, the function P *has the following properties.*

- P *is convex, non-increasing and differentiable throughout* $\mathbb{R}$.
- P *is real-analytic on the interval* $(-\infty, 1)$ *and is equal to* 0 *on* $[1, \infty)$.

Also, for the dimension function d *the following hold.*

- d *is continuous and strictly decreasing on* $[0, 2\log\gamma]$, *and vanishes on* $\mathbb{R} \setminus [0, 2\log\gamma)$.
- $d(0) := \lim_{t\searrow 0} -\widehat{P}(-t)/t = 1$, *and* $\lim_{t\nearrow 2\log\gamma} d'(t) = -\infty$.

3. Minkowski's question mark function

In this section we will investigate the relationships between the following two well known, elementary, measure theoretical dynamical systems.

The Farey-system $(\mathscr{U}, F, \nu_F)$: Let $F : \mathscr{U} \to \mathscr{U}$ refer to the *Farey-map* on $\mathscr{U}$, given by

$$F(x) := \begin{cases} x/(1-x) & \text{for} \quad 0 \le x \le 1/2, \\ (1-x)/x & \text{for} \quad 1/2 \le x \le 1. \end{cases}$$

One immediately verifies that the inverse branches of F are given by $f_1(x) = x/(x+1)$ and $f_2(x) = 1/(x+1)$. Also, let ν_F refer to the measure of maximal entropy of the system $(\mathscr{U}, F)$. That is, in particular, we have that $\nu_F(T_{n,k}) = 2^{-n}$, for all $n \in \mathbb{N}_0$ and $k = 0, \ldots, 2^n - 1$. Finally, note that ν_F is an F-invariant Gibbs measure for the potential function equal to some constant.

The tent-system $(\mathscr{U}, T, \nu_T)$: Let $T : [0,1] \to [0,1]$ refer to the *tent map* on $\mathscr{U}$, given by

$$T(x) := \begin{cases} 2x & \text{for} \quad 0 \le x \le 1/2, \\ 2 - 2x & \text{for} \quad 1/2 < x \le 1. \end{cases}$$

The measure of maximal entropy of the system $(\mathscr{U}, T)$ will be denoted by ν_T, and we clearly have that $\nu_T = \lambda$.

The following proposition shows that $(\mathscr{U},T)$ and $(\mathscr{U},F)$ are in fact topologically conjugate, and that the conjugating homeomorphism is given by the distribution function Δ_{ν_F} of the Farey measure ν_F. Moreover, we will see that Δ_{ν_F} is in fact equal to Q. Recall that Denjoy [5] [6] and Salem [30] showed that Q is given by

$$Q(x) = -2\sum_{k\in\mathbb{N}}(-1)^k\, 2^{-\sum_{i=1}^k a_i}, \text{ for all } x=[a_1,a_2,\ldots]\in\mathscr{U}. \tag{3.1}$$

The following commuting diagram summarises the statement of the proposition.

$$\begin{array}{ccc}
(\mathscr{U},\nu_F) & \xrightarrow{F} & (\mathscr{U},\nu_F)\\
{\scriptstyle \Delta_{\nu_F}=Q}\downarrow & & \downarrow{\scriptstyle \Delta_{\nu_F}=Q}\\
(\mathscr{U},\nu_T) & \xrightarrow{T} & (\mathscr{U},\nu_T)
\end{array}$$

Let us also remark that we believe that the proposition is well known to experts in this area. However, we were unable to locate it in the literature, and therefore decided to include the proof.

Proposition 3.1. *The two systems $(\mathscr{U},T)$ and $(\mathscr{U},F)$ are topologically conjugate, and the conjugating homeomorphism is given by the distribution function Δ_{ν_F} of the Farey-measure ν_F. Moreover, the function Δ_{ν_F} coincides with the Minkowski question mark function Q.*

Proof. Let us first show that Δ_{ν_F} and Q do in fact coincide. For this, let $x=[a_1,a_2,\ldots]\in\mathscr{U}$ be given. Recall that for the sequence $(p_k/q_k)_{k\in\mathbb{N}}$ of convergents of x (the sequence is finite if x is rational, and infinite otherwise) we have that $p_k/q_k=[a_1,\ldots,a_k]$, and that $x=\lim_{k\to\infty}p_k/q_k$. Clearly, the latter fact guarantees that it is sufficient to show that $\Delta_{\nu_F}(p_k/q_k)=Q(p_k/q_k)$, for each of the convergents of x. For this, we employ the following straightforward inductive argument. For ease of exposition, let $Q_k:=\Delta_{\nu_F}(p_k/q_k)$ and $A_k:=|Q_{k+1}-Q_{k-1}|$. For the start of the induction, note that if $a_1=1$ then $\Delta_{\nu_F}(1/a_1)=1=Q(1)$. Similarly, for $a_1>1$ we have

$$\Delta_{\nu_F}(1/a_1)=1-\sum_{i=1}^{a_1-1}2^{-i}=1-(1-2^{-(a_1-1)})=2\cdot 2^{-a_1}=Q(1/a_1).$$

For the inductive step, let us first state the following relations (which will be verified in what follows). For each $k\in\mathbb{N}\,(k\neq 1)$ we have

$$A_{k+1}=(1-2^{-a_{k+1}})\,|Q_k-Q_{k-1}|, \text{ and } Q_{k+1}=\begin{cases} Q_{k-1}+A_{k+1} & \text{for } k \text{ odd}\\ Q_{k-1}-A_{k+1} & \text{for } k \text{ even.}\end{cases} \tag{3.2}$$

The inductive assumption then is that $Q_i=Q(p_i/q_i)$ holds for each each $i\in\{1,\ldots,k\}$, for some $k\in\mathbb{N}$. Using this and (3.2), it follows for k odd,

$$\begin{aligned}
\Delta_{\nu_F}(p_{k+1}/q_{k+1}) &= Q_{k+1}=Q_{k-1}+A_{k+1}=Q_{k-1}+(1-2^{-a_{k+1}})\,|Q_k-Q_{k-1}|\\
&= Q_{k-1}+2\cdot 2^{-\sum_{i=1}^k a_i}(1-2^{-a_{k+1}})=Q_{k-1}+2\cdot 2^{-\sum_{i=1}^k a_i}-2\cdot 2^{-\sum_{i=1}^{k+1}a_i}\\
&= -2\sum_{m=1}^{k+1}(-1)^m\,2^{-\sum_{i=1}^m a_i}=Q(p_{k+1}/q_{k+1}).
\end{aligned}$$

Clearly, for k even one can argue almost in the same way, and this is left to the reader. This completes the inductive argument.

We now still have to prove the assertions in (3.2). We do this only for the case k even, and leave 'k odd' up to the reader. Recall that the interval bounded by p_{k-1}/q_{k-1} and p_{k+1}/q_{k+1} can be partitioned by the intermediate convergents $p_{k,m}/q_{k,m}$ of x. Here, $p_{k,m}/q_{k,m}$ is given by (see e.g. [20], see also Fig. 5.1 in the proof of Proposition 5.3)

$$p_{k,m} := m p_k + p_{k-1} \text{ and } q_{k,m} := m q_k + q_{k-1}, \text{ for all } m \in \{0, \ldots, a_{k+1}\}.$$

Then note that since $p_{k,1}/q_{k,1}$ is the mediant of p_k/q_k and p_{k-1}/q_{k-1}, it follows that $|Q(p_{k,1}/q_{k,1}) - Q(p_k/q_k)| = 2^{-1}|Q_k - Q_{k-1}|$. Likewise, $p_{k,2}/q_{k,2}$ is the mediant of $p_{k,1}/q_{k,1}$ and p_{k-1}/q_{k-1}, and hence $|Q(p_{k,2}/q_{k,1}) - Q(p_{k,1}/q_{k,1})| = 2^{-1}|Q_{k-1} - Q(p_{k,1}/q_{k,1})| = 2^{-2}|Q_k - Q_{k-1}|$. Clearly, this process can be continued until it terminates after a_{k+1} steps. In the final step we obtain the identity $|Q_{k+2} - Q(p_{k,a_{k+1}-1}/q_{k,a_{k+1}-1})| = 2^{-a_{k+1}}|Q_k - Q_{k-1}|$. The summation of these steps then gives

$$A_{k+1} = |Q_{k+2} - Q_k| = |Q_k - Q_{k-1}| \sum_{i=1}^{a_{k+1}} 2^{-i} = \left(1 - 2^{-a_{k+1}}\right)|Q_k - Q_{k-1}|.$$

This proves the first assertion in (3.2). The second assertion in (3.2) is an immediate consequence of the well known fact that the value of x is greater than any of its even-order convergents and is less than any of its odd-order convergents (see e.g. [20]). This finishes the proof of the equality of Δ_{ν_F} and Q.

For the proof of $T \circ \Delta_{\nu_F} = \Delta_{\nu_F} \circ F$, note that if $x = [a_1, a_2, \ldots]$ is such that $a_1 > 1$, then (3.1) gives

$$\begin{aligned} T\left(\Delta_{\nu_F}(x)\right) &= T\left(Q(x)\right) = 2\left(-2\sum_{k\in\mathbb{N}}(-1)^k\, 2^{-\sum_{i=1}^k a_i}\right) = -2\sum_{k\in\mathbb{N}}(-1)^k\, 2^{-\sum_{i=1}^k a_i - 1} \\ &= Q\left([a_1 - 1, a_2, \ldots]\right) = Q\left(x/(1-x)\right) = Q\left(F(x)\right) = \Delta_{\nu_F}\left(F(x)\right). \end{aligned}$$

Similar, for $x = [1, a_2, \ldots]$ we have

$$\begin{aligned} T\left(\Delta_{\nu_F}(x)\right) &= T\left(Q(x)\right) = 2 - 2\left(-2\sum_{k\in\mathbb{N}}(-1)^k\, 2^{-\sum_{i=1}^k a_i}\right) = -2\sum_{k\in\mathbb{N}}(-1)^k\, 2^{-\sum_{i=1}^k a_{i+1}} \\ &= Q\left((1-x)/x\right) = Q\left(F(x)\right) = \Delta_{\nu_F}\left(F(x)\right). \end{aligned}$$

Finally, the fact that Δ_{ν_F} is a homeomorphism is an immediate consequence of its construction. This finishes the proof. □

Remark 3.2. (1) An immediate implication of Proposition 3.1 is that $\nu_F = \nu_T \circ Q$, and that the measure theoretical and topological entropies $h_{\nu_F}(F)$, $h_{\nu_T}(T)$, $h_{\mathrm{top}}(T)$ and $h_{\mathrm{top}}(F)$ of both systems coincide and are equal to $h_{\mathrm{top}} := \log 2$. In fact, this also leads to an alternative proof of the fact that Q represents the distribution function of ν_F. Namely,

$$\nu_F([0,x)) = \nu_T \circ Q([0,x)) = \lambda \circ Q([0,x)) = Q(x), \text{ for each } x \in \mathscr{U}.$$

(2) Let us also remark that by the above, we immediately have that

$$Q(s_{n,k}/t_{n,k}) = k2^{-n}, Q(T_{n,k}) = D_{n,k}, \text{ and } \nu_F(T_{n,k}) = \lambda(Q(T_{n,k})) = 2^{-n}. \tag{3.3}$$

Also, the reader might like to recall that Q is related to the Stern-Brocot sequence $(s_{n,k}/t_{n,k})$ in the following way. We clearly have $Q(s_{0,0}/t_{0,0}) = 0$ and $Q(s_{0,1}/t_{0,1}) = 1$. Moreover, for two neighbours in the n-th Stern-Brocot sequence, we have

$$Q\left(\frac{s_{n,k}+s_{n,k+1}}{t_{n,k}+t_{n,k+1}}\right) = \frac{1}{2}\left(Q\left(\frac{s_{n,k}}{t_{n,k}}\right) + Q\left(\frac{s_{n,k+1}}{t_{n,k+1}}\right)\right).$$

Finally, recall that x is rational if and only if $Q(x)$ has a finite dyadic expansion, and that x is a quadratic surd if and only if $Q(x)$ is a rational number with an infinite dyadic expansion. In fact, the latter two properties of Q were Minkowski's original main motivation for introducing the function Q in the first place.

4. The integral of the Minkowski function w.r.t. the Gauss measure

The following proposition gives the main result of this section. For this recall that the Hausdorff dimension of a probability measure μ is given by (see e.g. [7])

$$\dim_H(\mu) := \inf\{\dim_H(X) : \mu(X) = 1\}.$$

Also, let $\mathbb{E}_\mu(\Delta_\nu) := \int \Delta_\nu \, d\mu$ refer to the μ-expectation of the distribution function $\Delta_\nu \in L^1(\mathscr{U}, \mu)$ of ν, for two probability measures ν and μ on $\mathscr{U}$. Moreover, let m_G refer to the Gauss measure. That is, m_G refers to the invariant measure of the Gauss map $G : x \mapsto 1/x \mod (1)$ absolutely continuous to λ.

Proposition 4.1. *For the m_G-expectation of Δ_{ν_F} and the ν_F-expectation of Δ_{m_G}, we have*

$$\mathbb{E}_{m_G}(Q) = \frac{\dim_H(\nu_F) - 1/2}{\dim_H(\nu_F)} \textit{ and } \mathbb{E}_{\nu_F}(\Delta_{m_G}) = \frac{1}{2\dim_H(\nu_F)}.$$

Proof. First note that the Stern-Brocot pressure function at zero corresponds to the Legendre transform $\widehat{P}$ at $-\chi_{\nu_F}$, where $\chi_{\nu_F} := \int \log|F'| \, d\nu_F$ denotes the Lyapunov exponent of F. That is,

$$\widehat{P}(-\chi_{\nu_F}) = \sup_{t\in\mathbb{R}}\{-t\cdot\chi_{\nu_F} - P(t)\} = -0\cdot\chi_{\nu_F} - P(0) = -h_{\mathrm{top}}.$$

Combining this observations with the fact that ν_F is the F-invariant Gibbs measure associated with $\mathscr{L}(\chi_{\nu_F})$, Theorem 2.1 implies

$$\dim_H(\nu_F) = \dim_H(\mathscr{L}(\chi_{\nu_F})) = -\widehat{P}(-\chi_{\nu_F})/\chi_{\nu_F} = h_{\mathrm{top}}/\chi_{\nu_F}. \tag{4.1}$$

Hence we are left with to determine χ_{ν_F} in terms of $\mathbb{E}_{\nu_F}(\Delta_{m_G})$. For this, recall that for the distribution function Δ_{m_G} of m_G we have

$$\Delta_{m_G}(x) := m_G([0,x)) = \int_0^x 1/(1+x)\, d\lambda(x)/h_{\mathrm{top}} = \log(1+x)/h_{\mathrm{top}}, \text{ for all } x \in \mathscr{U}.$$

Combining this with a straightforward computation of $|F'|$, one immediately verifies

$$\log|F'| = 2\,h_{\mathrm{top}} \cdot (\Delta_{m_G} \circ F)\,.$$

Hence, using the F-invariance of ν_F, it follows

$$\chi_{\nu_F} = \int \log|F'|\,d\nu_F = 2\,h_{\mathrm{top}} \int \Delta_{m_G} \circ F\,d\nu_F = 2\,h_{\mathrm{top}} \int \Delta_{m_G}\,d\nu_F = 2\,h_{\mathrm{top}}\,\mathbb{E}_{\nu_F}(\Delta_{m_G}).$$

By inserting this into (4.1) and solving for $\mathbb{E}_{\nu_F}(m_G)$, the second equality in the proposition follows. The first equality in the proposition is now an immediate consequence of the fact that

$$\mathbb{E}_{m_G}(\Delta_{\nu_F}) = 1 - \mathbb{E}_{\nu_F}(\Delta_{m_G}).$$

Since by Proposition 3.1 we have $\Delta_{\nu_F} = Q$, this finishes the proof. □

As an immediate consequence of Proposition 4.1 we obtain the following result of Kinney [21], which we state in its 'non-dynamical' form in which it was given in [21].

Corollary 4.2. *There exists a set $A \subset \mathscr{U}$ such that $\lambda(Q(A)) = 1$, and*

$$\dim_H(A) = \left(2\int_0^1 \log_2(1+x)\,dQ(x)\right)^{-1}.$$

Proof. Note that for the derivative $(f_i)'$ of the inverse branches of F we have

$$(f_i)'(x) = (1+x)^{-2}, \text{ for all } x \in \mathscr{U},\, i \in \{1,2\}.$$

Using this and the F-invariance of ν_F, it follows

$$\begin{aligned}
\chi_{\nu_F} &= \int \log|F'|\,d\nu_F = \int \left(\mathbf{1}_{[0,1/2)} \log|F' \circ f_1 \circ F| + \mathbf{1}_{[1/2,1]} \log|F' \circ f_2 \circ F|\right) d\nu_F \\
&= -\int \log|\left(F^{-1}\right)' \circ F|\,d\nu_F = -\int \log|\left(F^{-1}\right)'|\,d\nu_F \\
&= \int_{\mathscr{U}} \log\left((1+x)^2\right) d\nu_F(x).
\end{aligned}$$

Inserting this into (4.1), the result follows. □

Remark 4.3. Note that in [33] the numerical approximation $\dim_H(\nu_F) \approx 7/8$ was obtained (see also [22]). Hence, for the Stern-Brocot rate χ_{ν_F} associated with ν_F we have that $\chi_{\nu_F} = h_{\mathrm{top}}/\dim_H(\nu_F) \approx 0.792$, or in other words, $\ell(x) \approx 0.792$ for ν_F-almost every $x \in \mathscr{U}$. Moreover, this also immediately gives $\mathbb{E}_{m_G}(\Delta_{\nu_F}) \approx 3/7$ and $\mathbb{E}_{\nu_F}(\Delta_{m_G}) \approx 4/7$. (In fact, for the latter we derived, using numerical integration, the slightly better approximation $\mathbb{E}_{\nu_F}(\Delta_{m_G}) = 0.571612\ldots$).

Let us end this section by showing that the Hölder continuity of Q reflects precisely the range $[0, 2\log\gamma]$ of the Lyapunov spectrum associated with ℓ. For this, note that Salem showed in [30] that Q is $(\log 2/(2\log\gamma))$-Hölder continuous. That is,

$$|Q(x) - Q(y)| \ll |x-y|^{\log 2/(2\log\gamma)}, \text{ for all } x,y \in \mathscr{U}.$$

(Note that $\log 2/(2\log\gamma) \approx 0.7202$). As a consequence of this modulus of continuity of Q we have the following.

Lemma 4.4. *For each $x \in \mathscr{U}$, we have*

$$\limsup_{n\in\mathbb{N}} \ell_n(x) \leq 2\log\gamma.$$

Here, the constant $2\log\gamma \approx 0.9624$ is best possible, since it is attained for instance for each noble number, *that is a number whose continued fraction expansion eventually contains only 1's, and hence it is attained in particular for $x = \gamma^* := 1/\gamma$.*

Proof. The $(\log 2/(2\log\gamma))$-Hölder continuity of Q implies that for each $x \in \mathscr{U}$ and $n \in \mathbb{N}$, we have

$$\nu_F(T_n(x)) = \lambda\left(Q(T_n(x))\right) \ll \left(\lambda\left(T_n(x)\right)\right)^{\log 2/(2\log\gamma)}.$$

This implies, with $C > 0$ referring to some universal constant,

$$-n\log 2 = \log \nu_F(T_n(x)) \leq \frac{\log 2}{2\log\gamma}\log\lambda\left(T_n(x)\right) + C,$$

which gives

$$\limsup_{n\in\mathbb{N}} \ell_n(x) \leq 2\log\gamma.$$

For the remaining assertion recall that numerator and denominator of the n-th convergent $p_n/q_n := p_n(\gamma^*)/q_n(\gamma^*)$ of γ^* are equal to the n-th and $(n+1)$-th member of the Fibonacci sequence. That is,

$$p_n = \left(\gamma^n - (-\gamma^*)^n\right)/\sqrt{5} \text{ and } \; q_n = p_{n+1}.$$

Using this together with a well known Diophantine identity for continued fractions (see e.g. [20]), one immediately obtains, with $(O_{i,n})$ referring to certain sequences which tend to zero for n tending to infinity,

$$\begin{aligned}
|\gamma^* - p_n/q_n| &= \frac{1}{q_n^2(\gamma + p_n/q_n)} = \frac{1}{q_n^2(\sqrt{5}+O_{1,n})} = \frac{5}{\sqrt{5}+O_{1,n}}\left(\gamma^{n+1} - (-\gamma^*)^{n+1}\right)^{-2} \\
&= \frac{\sqrt{5}+O_{2,n}}{\gamma^2+O_{3,n}}\gamma^{-2n} = \left(\gamma^{-2}\sqrt{5}+O_{4,n}\right)\gamma^{-2n}.
\end{aligned}$$

Note that $Q(\gamma^*) = \sum_{i=0}^{\infty}(-2)^{-i}$ and $Q(p_n/q_n) = \sum_{i=0}^{n-1}(-2)^{-i}$, and hence, with $O_{5,n} := |\sum_{i=n+1}^{\infty}(-2)^{-i}|$,

$$|Q(\gamma^*) - Q(p_n/q_n)| = 2^{-n} - O_{5,n}.$$

Combining these two observations, it follows

$$\begin{aligned}
|Q(\gamma^*) - Q(p_n/q_n)| &= \left(\gamma^{-2n}\right)^{\log 2/(2\log\gamma)} - O_{5,n} \\
&= \left(\gamma^{-2}\sqrt{5}+O_{4,n}\right)^{-\log 2/(2\log\gamma)} |\gamma^* - p_n/q_n|^{\log 2/(2\log\gamma)} - O_{5,n}.
\end{aligned}$$

By taking logarithms, the result follows. $\square$

5. THE DERIVATIVE OF THE MINKOWSKI FUNCTION

Let us begin our analysis of the derivative of Q with the following lemma. Note that the instance in which either $Q'(x)$ exists or $Q'(x) = \infty$ will be referred to as $Q'(x)$ *exists in the generalised sense*.

Lemma 5.1. *For each $x \in \mathscr{U}$ we have that if $Q'(x)$ exists in the generalised sense, then*

$$Q'(x) = \lim_{n\to\infty} \frac{\nu_F(T_n(x))}{\lambda(T_n(x))}.$$

Proof. Let $x \in \mathscr{U}$ be given, and assume that $Q'(x)$ exists in the generalised sense. Let $T_n(x) = [s_{n,k}/t_{n,k}, s_{n,k+1}/t_{n,k+1})$ be the unique Stern-Brocot interval in $\mathscr{T}_n$ which contains x. Note that the alternating sum representation (3.1) of Q immediately gives that Q is a strictly increasing function. Using this, it follows that for each $n \in \mathbb{N}$ one of the following two cases has to occur. Firstly, if $Q(x)$ lies below or on the line through $Q(s_{n,k}/t_{n,k})$ and $Q(s_{n,k+1}/t_{n,k+1})$, then

$$\frac{Q(x) - Q(s_{n,k}/t_{n,k})}{x - s_{n,k}/t_{n,k}} \le \frac{Q(s_{n,k+1}/t_{n,k+1}) - Q(s_{n,k}/t_{n,k})}{s_{n,k+1}/t_{n,k+1} - s_{n,k}/t_{n,k}} \le \frac{Q(s_{n,k+1}/t_{n,k+1}) - Q(x)}{s_{n,k+1}/t_{n,k+1} - x}.$$

Secondly, if $Q(x)$ lies above or on the line through $Q(s_{n,k}/t_{n,k})$ and $Q(s_{n,k+1}/t_{n,k+1})$, then

$$\frac{Q(s_{n,k+1}/t_{n,k+1}) - Q(x)}{s_{n,k+1}/t_{n,k+1} - x} \le \frac{Q(s_{n,k+1}/t_{n,k+1}) - Q(s_{n,k}/t_{n,k})}{s_{n,k+1}/t_{n,k+1} - s_{n,k}/t_{n,k}} \le \frac{Q(x) - Q(s_{n,k}/t_{n,k})}{x - s_{n,k}/t_{n,k}}.$$

Hence, by taking the limit for n tending to infinity, and noting that

$$Q(s_{n,k+1}/t_{n,k+1}) - Q(s_{n,k}/t_{n,k}) = \nu_F([0, s_{n,k+1}/t_{n,k+1})) - \nu_F([0, s_{n,k}/t_{n,k})) = \nu_F(T_n(x)),$$

the assertion follows. □

The following result was obtained in [26] using continued fraction expansions. Here, we give an alternative proof which uses Stern-Brocot sequences, and which appears to us to be far more canonical than the one given in [26].

Lemma 5.2. *For each $x \in \mathscr{U}$ we have that if $Q'(x)$ exists in the generalised sense, then*

$$Q'(x) \in \{0, \infty\}.$$

Proof. Let $x \in \mathscr{U}$ be given such that $Q'(x)$ exists in the generalised sense. Without loss of generality we can assume that x is irrational. By Lemma 5.1, we then have

$$Q'(x) = \lim_{n\to\infty} \frac{\nu_F(T_n(x))}{\lambda(T_n(x))}.$$

Let us assume by way of contradiction that $Q'(x) = c$, for some $0 < c < \infty$. Since we have $Q'(x) = \lim_{n\to\infty} 2^{-n}/\lambda(T_n(x))$, it follows that

$$\lim_{n\to\infty} \frac{2^n \lambda(T_n(x))}{2^{n+1} \lambda(T_{n+1}(x))} = 1,$$

and hence,

$$\lim_{n\to\infty}\frac{\lambda\left(T_n(x)\right)}{\lambda\left(T_{n+1}(x)\right)}=2. \tag{5.1}$$

In order to proceed, let $T_n(x)=[s_{n,k}/t_{n,k},s_{n,k+1}/t_{n,k+1})$, and assume that there is a 'type-change' at $T_n(x)$. That is, assume that $T_{n-1}(x)=[(s_{n,k}-s_{n,k+1})/(t_{n,k}-t_{n,k+1}),s_{n,k+1}/t_{n,k+1})$ and $T_{n+1}(x)=[s_{n,k}/t_{n,k},(s_{n,k}+s_{n,k+1})/(t_{n,k}+t_{n,k+1}))$. We then immediately obtain

$$\frac{\lambda\left(T_n(x)\right)}{\lambda\left(T_{n+1}(x)\right)}=\frac{s_{n,k+1}/t_{n,k+1}-s_{n,k}/t_{n,k}}{(s_{n,k}+s_{n,k+1})/(t_{n,k}+t_{n,k+1})-s_{n,k}/t_{n,k}}=\frac{t_{n,k}(t_{n,k}+t_{n,k+1})}{t_{n,k}t_{n,k+1}}=1+\frac{t_{n,k}}{t_{n,k+1}}.$$

Combining this with (5.1), it follows

$$\lim_{n\to\infty}\frac{t_{n,k}}{t_{n,k+1}}=1. \tag{5.2}$$

By considering the quotient of $\lambda\left(T_{n-1}(x)\right)$ and $\lambda\left(T_n\left(x\right)\right)$, a similar computation gives

$$\frac{\lambda\left(T_{n-1}(x)\right)}{\lambda\left(T_n(x)\right)}=\frac{t_{n,k}}{t_{n,k}-t_{n,k+1}}=\frac{1}{1-t_{n,k+1}/t_{n,k}}. \tag{5.3}$$

Then observe that since x is irrational, there have to be infinitely many type-changes in $\{T_n(x):n\in\mathbb{N}\}$. That is, there exist sequences $(n_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$ such that $T_{n_i}(x)=[s_{n_i,k_i}/t_{n_i,k_i},s_{n_i,k_i+1}/t_{n_i,k_i+1})$, and such that there is a type-change at $T_{n_i}(x)$ for each $i\in\mathbb{N}$. Therefore, combining this with (5.2) and (5.3), it now follows

$$\lim_{i\to\infty}\frac{\lambda(T_{n_i-1}(x))}{\lambda\left(T_{n_i}(x)\right)}=\lim_{i\to\infty}\frac{1}{1-t_{n_i,k_i+1}/t_{n_i,k_i}}=\infty.$$

This contradicts (5.1), and hence finishes the proof of the lemma. □

The following proposition will turn out to be crucial in the multifractal analysis to come. For ease of exposition, we let $[x,y)_\pm$ refer to the interval bounded by x and y. That is, $[x,y)_\pm:=[x,y)$ if $x\leq y$, and $[x,y)_\pm:=[y,x)$ if $x\geq y$.

Proposition 5.3. *For $x=[a_1,a_2,\ldots]\in\mathscr{U}$ and with p_k/q_k referring to the k-th convergent of x, the following hold.*

(i)
$$\textit{If}\ \lim_{k\to\infty}\frac{\nu_F\left([p_k/q_k,p_{k+1}/q_{k+1})_\pm\right)}{\lambda\left([p_k/q_k,p_{k+1}/q_{k+1})_\pm\right)}=\infty,\ \textit{then}\ Q'(x)=\infty.$$

(ii)
$$\textit{If}\ \lim_{k\to\infty}a_{k+1}\cdot\frac{\nu_F\left([p_k/q_k,p_{k+1}/q_{k+1})_\pm\right)}{\lambda\left([p_k/q_k,p_{k+1}/q_{k+1})_\pm\right)}=0,\ \textit{then}\ Q'(x)=0.$$

Proof. Let $x=[a_1,a_2,\ldots]\in\mathscr{U}$ be given as stated in (i). Using (3.1) and the fact that $|p_kq_{k+1}-p_{k+1}q_k|=1$, we immediately obtain

$$\frac{\nu_F\left([p_k/q_k,p_{k+1}/q_{k+1})_\pm\right)}{\lambda\left([p_k/q_k,p_{k+1}/q_{k+1})_\pm\right)}=\frac{|Q(p_k/q_k)-Q(p_{k+1}/q_{k+1})|}{|p_k/q_k-p_{k+1}/q_{k+1}|}=\frac{2q_kq_{k+1}}{2^{\sum_{i=1}^{k+1}a_i}}. \tag{5.4}$$

Before we proceed, let us first recall that the intermediate convergents $p_{k,m}/q_{k,m}$ of x are given by (see e.g. [20])

$$p_{k,m} := mp_k + p_{k-1} \text{ and } q_{k,m} := mq_k + q_{k-1}, \text{ for all } m \in \{0, \ldots, a_{k+1}\}.$$

Since $p_{k,0}/q_{k,0} = p_{k-1}/q_{k-1} = [a_1, \ldots, a_{k-1}]$, $p_{k,a_{k+1}}/q_{k,a_{k+1}} = p_{k+1}/q_{k+1}$ and $p_{k,n}/q_{k,n} = [a_1, \ldots, a_k, n]$ for $n \in \{1, \ldots, a_{k+1}\}$, we immediately obtain from (3.1) that for each $m \in \{0, \ldots, a_{k+1} - 1\}$,

$$|Q(x) - Q(p_{k,m}/q_{k,m})| \gg 2^{-(m+\sum_{j=1}^{k} a_j)},$$

and

$$|Q(x) - Q(p_{k,a_{k+1}}/q_{k,a_{k+1}})| \gg 2^{-\sum_{j=1}^{k+2} a_j}.$$

We then compute for $m \in \{0, \ldots, a_{k+1}\}$, with $r_n := [a_n; a_{n+1}, \ldots]$ referring to the n-th remainder of x,

$$|x - p_{k,m}/q_{k,m}| \quad = \quad \left| \frac{r_{k+1}p_k + p_{k-1}}{r_{k+1}q_k + q_{k-1}} - \frac{mp_k + p_{k-1}}{mq_k + q_{k-1}} \right| = \frac{r_{k+1} - m}{(r_{k+1}q_k + q_{k-1})(mq_k + q_{k-1})}$$

Now, let $y \in \mathscr{U}$ be fixed such that $y > x$. Then there exist $k \in \mathbb{N}$ and $m \in \{0, \ldots, a_{k+1} - 1\}$ such that $p_{k,m+1}/q_{k,m+1} < y \leq p_{k,m}/q_{k,m}$. For each $m \in \{0, \ldots, a_{k+1} - 2\}$, we then have

$$\begin{aligned}
\frac{Q(y) - Q(x)}{y - x} \quad &\geq \quad \frac{Q(p_{k,m+1}/q_{k,m+1}) - Q(x)}{p_{k,m}/q_{k,m} - x} \\
&\gg \quad \frac{(r_{k+1}q_k + q_{k-1})(mq_k + q_{k-1})}{q_k q_{k+1}(r_{k+1} - m)} \frac{q_k q_{k+1}}{2^{(m+1)+\sum_{j=1}^{k} a_j}} \\
&= \quad \frac{2^{a_{k+1}-(m+1)}(mq_k + q_{k-1})(r_{k+1}q_k + q_{k-1})}{(r_{k+1} - m)q_k q_{k+1}} \frac{q_k q_{k+1}}{2^{\sum_{j=1}^{k+1} a_j}} \\
&\gg \quad \frac{q_k q_{k+1}}{2^{\sum_{j=1}^{k+1} a_j}}.
\end{aligned}$$

Note that the latter argument does not work for $m = a_{k+1} - 1$. In this case, that is for $p_{k+1}/q_{k+1} < y \leq p_{k,a_{k+1}-1}/q_{k,a_{k+1}-1}$, we have to consider the partition of the interval $(p_{k+1}/q_{k+1}, p_{k,a_{k+1}-1}/q_{k,a_{k+1}-1}]$ obtained from what we call the 'micro-intermediate convergents' $\widehat{p}_{k,n}/\widehat{q}_{k,n}$ (cf. Fig. 5.1). These are given for $n \in \mathbb{N}$ by

$$\widehat{p}_{k,n} := np_{k+1} - p_k \text{ and } \widehat{q}_{k,n} := nq_{k+1} - q_k.$$

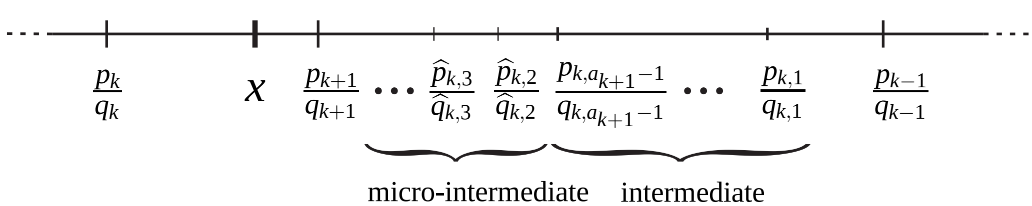

FIGURE 5.1. Regular, intermediate, and micro-intermediate convergents for $x \in \mathscr{U}$ and $k \in \mathbb{N}$ even.

Note that $\widehat{p}_{k,1}/\widehat{q}_{k,1} = ((a_{k+1}-1)p_k + p_{k-1})/((a_{k+1}-1)q_k + q_{k-1}) = p_{k,a_{k+1}-1}/q_{k,a_{k+1}-1}$. Also, one immediately verifies that the continued fraction expansion of $\widehat{p}_{k,n}/\widehat{q}_{k,n}$ is given by

$$\widehat{p}_{k,n}/\widehat{q}_{k,n} = [a_1, \ldots, a_k, a_{k+1}-1, 1, n].$$

Clearly, if $y \in (p_{k+1}/q_{k+1}, p_{k,a_{k+1}-1}/q_{k,a_{k+1}-1}]$, then there exists $l \in \mathbb{N}$ such that $\widehat{p}_{k,l+1}/\widehat{q}_{k,l+1} < y \leq \widehat{p}_{k,l}/\widehat{q}_{k,l}$. Using (3.1) together with the fact that Q is strictly increasing, one then immediately obtains the estimate

$$\begin{aligned} Q(y) - Q(x) &\geq Q(\widehat{p}_{k,l+1}/\widehat{q}_{k,l+1}) - Q(x) \gg 2^{-\sum_{i=1}^{k+1} a_i}\left(1 - 2^{-(l+1)} - 2^{-a_{k+2}}\right) \\ &\geq 2^{-\sum_{i=1}^{k+1} a_i}. \end{aligned}$$

Furthermore, in this situation we trivially have

$$y - x \leq p_{k,a_{k+1}-1}/q_{k,a_{k+1}-1} - p_k/q_k \ll 1/(q_k q_{k+1}).$$

Hence, this shows that also in this case we have

$$\frac{Q(y)-Q(x)}{y-x} \gg \frac{q_k q_{k+1}}{2^{\sum_{j=1}^{k+1} a_j}}.$$

Combining the above with (5.4) and the assumption in (i), it now follows

$$\lim_{y \to x+} \frac{|Q(x)-Q(y)|}{|x-y|} \gg \lim_{k\to\infty} \frac{q_{2k}q_{2k+1}}{2^{\sum_{j=1}^{2k+1} a_j}} = \lim_{k\to\infty} \frac{\nu_F\left([p_{2k}/q_{2k}, p_{2k+1}/q_{2k+1})\right)}{\lambda\left([p_{2k}/q_{2k}, p_{2k+1}/q_{2k+1})\right)} = \infty.$$

Clearly, a minor modification of the argument above then also gives that for the limit from the left we have

$$\lim_{y \to x-} \frac{|Q(x)-Q(y)|}{|x-y|} \gg \lim_{k\to\infty} \frac{q_{2k-1}q_2}{2^{\sum_{j=1}^{2k} a_j}} = \lim_{k\to\infty} \frac{\nu_F\left([p_{2k-1}/q_{2k-1}, p_{2k}/q_{2k})\right)}{\lambda\left([p_{2k-1}/q_{2k-1}, p_{2k}/q_{2k})\right)} = \infty.$$

Hence, we conclude that $Q'(x) = \infty$, and this finishes the proof of the assertion in (i).

For the proof of (ii), we proceed similar as for (i). Namely, let $y \in \mathscr{U}$ be fixed such that $y > x$. Then there exist $k \in \mathbb{N}$ and $m \in \{0, \ldots, a_{k+1}-1\}$ such that $p_{k,m+1}/q_{k,m+1} < y \leq p_{k,m}/q_{k,m}$. For each $m \in \{0, \ldots, a_{k+1}-2\}$, we then have

$$\begin{aligned} \frac{Q(y)-Q(x)}{y-x} &\ll \frac{Q(p_{k,m}/q_{k,m}) - Q(x)}{p_{k,m+1}/q_{k,m+1} - x} \\ &\ll \frac{(r_{k+1}q_k + q_{k-1})((m+1)q_k + q_{k-1})}{(r_{k+1} - (m+1))q_{k-1}q_k} \frac{q_{k-1}q_k}{2^{m+\sum_{j=1}^{k} a_j}} \\ &\ll \frac{a_{k+1}(m+1)q_k^2}{2^m(a_{k+1} - (m+1))q_{k-1}q_k} \frac{q_{k-1}q_k}{2^{\sum_{j=1}^{k} a_j}} \\ &\ll \frac{a_{k+1}(m+1)q_k}{2^m(a_{k+1} - (m+1))} \cdot a_k \cdot \frac{q_{k-1}q_k}{2^{\sum_{j=1}^{k} a_j}} \\ &\ll a_k \cdot \frac{\nu_F\left([p_{k-1}/q_{k-1}, p_k/q_k)_\pm\right)}{\lambda\left([p_{k-1}/q_{k-1}, p_k/q_k)_\pm\right)}. \end{aligned}$$

For the remaining case $m = a_{k+1} - 1$, we observe

$$(Q(y) - Q(x))/(y - x) \leq Q(p_{k,a_{k+1}-1}/q_{k,a_{k+1}-1}) - Q(x) \ll 2^{-\sum_{j=1}^{k+1} a_j},$$

and

$$y - x \geq 1/(2q_k q_{k+1}).$$

Therefore, also in this case we have

$$\frac{Q(y) - Q(x)}{y - x} \ll \frac{q_k q_{k+1}}{2^{\sum_{j=1}^{k+1} a_j}} \leq a_{k+1} \cdot \frac{\nu_F\left([p_k/q_k, p_{k+1}/q_{k+1})_\pm\right)}{\lambda\left([p_k/q_k, p_{k+1}/q_{k+1})_\pm\right)}.$$

A similar estimate can be given for $y < x$, and this is left to the reader. Clearly, using the assumption in (ii), we can now proceed as in the proof of (i), and this then gives $Q'(x) = 0$. This completes the proof of the proposition. □

Remark 5.4. Note that the proof of Proposition 5.3 also shows that the following implication holds.

$$\limsup_{k\to\infty} \frac{\nu_F\left([p_k/q_k, p_{k+1}/q_{k+1})\right)}{\lambda\left([p_k/q_k, p_{k+1}/q_{k+1})\right)} = \infty \implies \limsup_{y\to x} \frac{Q(x) - Q(y)}{x - y} = \infty.$$

Moreover, note that since

$$|Q(x) - Q(p_k/q_k)| \leq 2^{1-\sum_{i=1}^{k+1} a_i} \text{ and } |x - p_k/q_k| \geq \frac{1}{2q_k q_{k+1}}, \text{ for all } k \in \mathbb{N},$$

we also have the implication:

$$\liminf_{k\to\infty} \frac{\nu_F\left([p_k/q_k, p_{k+1}/q_{k+1})\right)}{\lambda\left([p_k/q_k, p_{k+1}/q_{k+1})\right)} = 0 \implies \liminf_{y\to x} \frac{Q(x) - Q(y)}{x - y} = 0.$$

For later use, let us also state the following immediate corollary.

Corollary 5.5. *For $x \in \mathscr{U}$ we have*

$$Q'(x) = \infty \textit{ if and only if } \lim_{n\to\infty} \nu_F(T_n(x))/\lambda\left(T_n(x)\right) = \infty.$$

Proof. The 'only if part' of the corollary was obtained in Lemma 5.1. By noting that the sequence $\left(\nu_F\left([p_k/q_k, p_{k+1}/q_{k+1})\right)/\lambda\left([p_k/q_k, p_{k+1}/q_{k+1})\right)\right)_{k\in\mathbb{N}}$ is a subsequence of $\left(\nu_F(T_n(x))/\lambda\left(T_n(x)\right)\right)_{n\in\mathbb{N}}$, the 'if part' of the corollary is an immediate consequence of Proposition 5.3. Here, p_k/q_k refers once more to the k-th convergent of x. □

6. Fractal analysis of the derivative of the Minkowski function

By Lemma 5.2, the result of [26] respectively, the unit interval can be decomposed into pairwise disjoint sets as follows.

$$\mathscr{U} = \Lambda_0 \cup \Lambda_\infty \cup \Lambda_\sim,$$

where

$$\Lambda_\theta := \{x \in \mathscr{U} : Q'(x) = \theta\} \text{ for } \theta \in \{0, \infty\}, \quad \text{and } \Lambda_\sim := \mathscr{U} \setminus (\Lambda_0 \cup \Lambda_\infty).$$

Clearly, by Lemma 5.2 we have that $\Lambda_\sim = \{x \in \mathscr{U} : Q'(x)\,\text{does not exist and}\; Q'(x) \neq \infty\}$.
Let us begin our analysis of this decomposition with the following result.

Proposition 6.1. *For $s \in (h_{\mathrm{top}}, 2\log\gamma]$ we have*

$$\mathscr{L}(s) \subset \Lambda_\infty.$$

Whereas, for $s \in [0, h_{\mathrm{top}})$ we have

$$\mathscr{L}(s) \subset \Lambda_0.$$

Proof. Let $x \in \mathscr{L}(s)$ be given. By definition of $\mathscr{L}(s)$, we then have

$$\lim_{n\to\infty} \ell_n(x) = s.$$

Hence, for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$n(s-\varepsilon) \leq \log\left(1/\lambda\left(T_n(x)\right)\right) \leq n(s+\varepsilon), \quad \text{for all} \quad n \geq N_\varepsilon.$$

From this we immediately deduce that

$$e^{-n(s+\varepsilon-h_{\mathrm{top}})} \leq 2^n\,\lambda\left(T_n(x)\right) \leq e^{-n(s-\varepsilon-h_{\mathrm{top}})}, \quad \text{for all} \quad n \geq N_\varepsilon. \tag{6.1}$$

For $s \in (h_{\mathrm{top}}, 2\log\gamma]$, this implies $\lim_{n\to\infty} \lambda\left(T_n(x)\right)/\nu_F\left(T_n(x)\right) = \lim_{n\to\infty} 2^n\,\lambda\left(T_n(x)\right) = 0$. By Corollary 5.5, we then have that $Q'(x) = \infty$, and hence $x \in \Lambda_\infty$. This finishes the proof of the first part of the proposition.
For the second part, let $s \in [0, h_{\mathrm{top}})$ and $x = [a_1, a_2, \ldots] \in \mathscr{L}(s)$ be fixed. Let q_n refer to the denominator of the n-th convergent $p_n/q_n := [a_1, a_2, \ldots, a_n]$ of x. We then have

$$\lim_{n\to\infty} \frac{\log(a_n q_n q_{n-1})}{\sum_{j=1}^n a_j} = \lim_{n\to\infty} \frac{\log(q_n q_{n-1})}{\sum_{j=1}^n a_j} = \lim_{n\to\infty} \ell_n(x) < h_{\mathrm{top}}.$$

Here, the last equality is a consequence of [19, Proposition 2.1]. Similar to the above, a straight forward calculation then shows that $\lim_{n\to\infty}(a_n q_n q_{n-1})/2^{\sum_{j=1}^n a_j} = 0$. Using the second part of Proposition 5.3, it follows $Q'(x) = 0$. □

Note that an immediate consequence of Proposition 6.1 is that the essential support of ν_F is contained in Λ_∞. Moreover, by combining Proposition 6.1 and Remark 5.4 we immediately obtain the following corollary. Here, q_n refers once more to the denominator of the n-th convergent $p_n/q_n := [a_1, a_2, \ldots, a_n]$ of $x = [a_1, a_2, \ldots]$.

Corollary 6.2. *For $x \in \mathscr{U}$ the following hold.*

(i) *If* $\lim_{n\to\infty} \frac{1}{2\log q_n} \sum_{i=1}^n a_i < 1/h_{\mathrm{top}}$, *then* $x \in \Lambda_\infty$.
(ii) *If* $\lim_{n\to\infty} \frac{1}{2\log q_n} \sum_{i=1}^n a_i > 1/h_{\mathrm{top}}$, *then* $x \in \Lambda_0$.
(iii) *If* $\limsup_{n\to\infty} \frac{1}{2\log q_n} \sum_{i=1}^n a_i > 1/h_{\mathrm{top}}$ *and* $\liminf_{n\to\infty} \frac{1}{2\log q_n} \sum_{i=1}^n a_i < 1/h_{\mathrm{top}}$, *then* $x \in \Lambda_\sim$.

Remark 6.3. Note that a similar type of result was obtained in [26]. Namely, on the basis of the assumption that $Q'(x)$ exists in the generalised sense, the following hold.

(i) *If* $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i < 2\log\gamma\,(=1.3884\ldots)$, *then* $x \in \Lambda_\infty$.
(ii) *If* $\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i > \rho\ (=5.3197\ldots)$, *then* $x \in \Lambda_0$.
(Here, ρ *is given implicitly by* $(1+\rho)^{1/\rho} = \sqrt{2}$*).*

For the following proposition, let $N : \mathscr{U} \to \mathbb{N}$ be given by $N([a_1,a_2,\ldots]) := a_1$, and let $I : \mathscr{U} \to \mathbb{R}$ refer to the potential function which is given by $I(x) := \log|G'(x)|$, with G denoting the Gauss map. For $0 < s < t < \infty$, we then define the sets

$$\mathscr{L}^*(s) := \left\{ x \in \mathscr{U} : \limsup_{n\to\infty} \frac{S_n I(x)}{S_n N(x)} \geq s \right\},\ \mathscr{L}_*(s) := \left\{ x \in \mathscr{U} : \liminf_{n\to\infty} \frac{S_n I(x)}{S_n N(x)} \geq s \right\},$$
$$\mathscr{L}(s,t) := \left\{ x \in \mathscr{U} : \liminf_{n\to\infty} \frac{S_n I(x)}{S_n N(x)} \leq s, \limsup_{n\to\infty} \frac{S_n I(x)}{S_n N(x)} \geq t \right\},$$

where $S_n\phi(x) := \sum_{k=0}^{n-1} \phi(G^k(x))$ refers to the n-th Birkhoff sum of a function ϕ. Moreover, for $x = [a_1,a_2,\ldots] \in \mathscr{U}$ and $n \in \mathbb{N}$, we use the notation $C_n(x) := \{[b_1,b_2,\ldots] \in \mathscr{U} : b_i = a_i, \text{for all } i \in \{1,\ldots,n\}\}$ to denote the unique n-cylinder containing x.

Proposition 6.4.

(i) *For each* $s \in [0, 2\log\gamma]$, *we have*

$$\dim_H(\mathscr{L}_*(s)) = \dim_H(\mathscr{L}^*(s)) = \dim_H(\mathscr{L}(s)).$$

(ii) *For each* $0 < s_0 \leq s_1 \leq 2\log\gamma$, *we have*

$$\dim_H(\mathscr{L}(s_0,s_1)) = \dim_H(\mathscr{L}(s_1)).$$

Proof. ad (i). The inequality $\dim_H(\mathscr{L}_*(s)) \leq \dim_H(\mathscr{L}^*(s))$ follows immediately from $\mathscr{L}_*(s) \subset \mathscr{L}^*(s)$. For the proof of the upper estimate $\dim_H(\mathscr{L}^*(s)) \leq -\widehat{P}(-s)/s$ we refer to [19, Lemma 5.4]. Note that in [19] we in fact considered the set $\mathscr{L}_*(s)$, rather than the set $\mathscr{L}^*(s)$. However, one immediately sees that in the proof of [19, Lemma 5.4] '$\liminf$' can be replaced by '$\limsup$'. Using Theorem 2.1 and the fact that $\mathscr{L}(s) \subset \mathscr{L}_*(s)$, then gives rise to the statement in (i).

ad (ii). Since "$\leq$" is a direct consequence of (i), we only have to show "$\geq$". Using standard techniques from geometric measure theory (cf. e.g. [24]), it is sufficient to show that there exists a probability measure μ such that

(A) $\mu(\mathscr{L}(s_0,s_1)) > 0$,
(B) $\liminf_{n\to\infty} \frac{-\log\mu(C_n(x))}{S_n I(x)} \geq \dim_H(\mathscr{L}(s_1))$, for μ-almost every $x \in \mathscr{U}$.

For this, let us first recall the following outcome of the thermodynamic formalism of [19]. For $i = 0,1$, let μ_i be the Gibbs measures on $\mathscr{U}$ for the potential function $-P(t(s_i))I - t(s_i)N$, for P denoting the pressure function defined in (2.1), and t the inverse function of P' (we refer to [19, Proposition

4.2] for the details). For these measures it was shown in [19] that $\int I\,d\mu_i / \int N\,d\mu_i = s_i$, $h_{\mu_i} / \int I\,d\mu_i = \dim_H(\mu_i) = \dim_H(\mathscr{L}(s_i))$, as well as

$$\lim_{n\to\infty} \frac{S_n I(x)}{n} = \int I\,d\mu_i \in (0,\infty) \text{ and } \lim_{n\to\infty} \frac{-\log \mu_i(C_n(x))}{n} = h_{\mu_i}, \quad \text{for } \mu_i\text{-almost every } x \in \mathscr{U}. \quad (6.2)$$

For ease of exposition, let us put $\theta(k) :\equiv k \mod (2)$. Using Egorov's Theorem, it follows that there exists an increasing sequence $(m_k)_{k\in\mathbb{N}}$ and a sequence $(\Gamma_k)_{k\in\mathbb{N}}$ of Borel subsets of $\mathscr{U}$, such that we have $\mu_{\theta(k)}(\Gamma_k) \geq 1 - 2^{-(k+1)}$, and such that for all $x \in \Gamma_k$ and $n \geq m_k$,

- $\left| \frac{S_n I(x)}{n} - \int I\,d\mu_{\theta(k)} \right| < k^{-1}$,
- $\left| \frac{-\log \mu_{\theta(k)}(C_n(x))}{n} - h_{\mu_{\theta(k)}} \right| < k^{-1}$,
- $\frac{-\log \mu_{\theta(k)}(C_n(x))}{S_n I(x)} > \dim_H\left(\mu_{\theta(k)}\right) - k^{-1}$.

Define $n_0 := 1 + 1/m_1$ and let $n_k := \prod_{i=1}^k (m_i + 1)$, for each $k \in \mathbb{N}$. Then define the countable family of cylinder sets

$$\mathscr{C}_k := \left\{ C_{n_{k-1}m_k}(x) : x \in \Gamma_k \right\}, \text{ for each } k \in \mathbb{N}.$$

This allows to introduce another family $(\mathscr{D}_k)_{k\in\mathbb{N}}$ of cylinder sets as follows. Let $\mathscr{D}_1 := \mathscr{C}_1$, and for $k \geq 2$ define

$$\mathscr{D}_k := \{ CD : C \in \mathscr{D}_{k-1}, D \in \mathscr{C}_k \},$$

where CD denotes the concatenation of the cylinders C and D. By construction, we have that each cylinder set in $\mathscr{D}_k$ has length equal to n_k, for each $k \in \mathbb{N}$. We can then define the set

$$\mathscr{M} := \bigcap_{k\in\mathbb{N}} \bigcup_{D\in\mathscr{D}_k} D.$$

One immediately verifies that $\mathscr{M}$ is non-empty. Next, using Kolmogorov's consistency theorem, we define the Cantor-measure m on $\mathscr{M}$, by setting $m(C) := \mu_1(C)$ if $C \in \mathscr{D}_1$, and for $C = D'C' \in \mathscr{D}_k$ such that $D' \in \mathscr{D}_{k-1}$ and $C' \in \mathscr{C}_k$, we let

$$m(C) := m(D')\mu_{\theta(k)}(C').$$

Clearly, m admits an extension μ to $\mathscr{U}$, and this is given by $\mu(A) := m(A \cap \mathscr{M})$, for each $A \subset \mathscr{U}$ measurable. By construction we then have that

$$\mu(\mathscr{M}) \geq \prod_{k\in\mathbb{N}} \left(1 - 2^{-k}\right) > 0.$$

Since I is Hölder continuous, we obtain for $x \in C \in \mathscr{D}_k$,

$$\left| \frac{S_{n_k} I(x)}{n_k} \right| \quad \leq \quad \frac{1}{m_k+1} \left| \frac{1}{n_{k-1}} S_{n_{k-1}} I(x) \right| + \frac{m_k}{m_k+1} \left| \frac{1}{n_{k-1}m_k} S_{n_{k-1}m_k} I\left(G^{n_{k-1}}x\right) \right|.$$

Using this, a straightforward inductive argument then gives that $S_{n_k}I(x)/n_k$ is bounded, and hence,

$$\lim_{k\to\infty}\left|\frac{S_{n_k}I(x)}{n_k}-\int I\,d\mu_{\theta(k)}\right|=0.$$

This shows that $\mathscr{M}\subset\mathscr{L}(s_0,s_1)$, and thus the assertion in (A) follows.

For the proof of (B), first note that an argument similar to the one just given, shows

$$\lim_{k\to\infty}\left|\frac{-\log\left(\mu\left(C_{n_k}(x)\right)\right)}{n_k}-h_{\mu_{\theta(k)}}\right|=0.$$

Then note that $C_{n_k}(x)=C_{n_{k-1}}(x)C_{m_kn_{k-1}}(G^{n_{k-1}}x)$, for each $x\in\mathscr{M}$ and $k\in\mathbb{N}$. Using this, it follows

$$\begin{aligned}
&\frac{-\log\left(\mu\left(C_{n_k}(x)\right)\right)}{S_{n_k}I(x)}\\
&= \frac{n_{k-1}}{n_k}\frac{\frac{S_{n_{k-1}}I(x)}{n_{k-1}}}{\frac{S_{n_k}I(x)}{n_k}}\cdot\frac{-\log\left(\mu\left(C_{n_{k-1}}(x)\right)\right)}{n_{k-1}}+\frac{m_kn_{k-1}}{n_k}\frac{\frac{S_{m_kn_{k-1}}I(G^{n_{k-1}}x)}{m_kn_{k-1}}}{\frac{S_{n_k}I(x)}{n_k}}\cdot\frac{-\log\left(\mu_{\theta(k)}\left(C_{m_kn_{k-1}}\left(G^{n_{k-1}}x\right)\right)\right)}{S_{m_kn_{k-1}}I\left(G^{n_{k-1}}x\right)}\\
&= \frac{1}{m_k+1}\underbrace{\frac{S_{n_{k-1}}I(x)/n_{k-1}}{S_{n_k}I(x)/n_k}\cdot\frac{-\log\left(\mu\left(C_{n_{k-1}}(x)\right)\right)}{n_{k-1}}}_{\text{bounded}}\\
&\qquad+\frac{m_k}{m_k+1}\underbrace{\frac{\frac{S_{m_kn_{k-1}}I(G^{n_{k-1}}x)}{m_kn_{k-1}}}{\frac{S_{n_k}I(x)}{n_k}}}_{\to 1}\cdot\frac{-\log\left(\mu_{\theta(k)}\left(C_{m_kn_{k-1}}\left(G^{n_{k-1}}x\right)\right)\right)}{S_{m_kn_{k-1}}I\left(G^{n_{k-1}}x\right)}.
\end{aligned}$$

This implies that

$$\liminf_{k\to\infty}\frac{-\log\left(\mu\left(C_{n_k}(x)\right)\right)}{S_{n_k}I(x)}\geq\dim_H(\mu_1). \tag{6.3}$$

Also, for $n_k\leq n<n_k+m_k$ we immediately obtain

$$\frac{-\log\left(\mu\left(C_n(x)\right)\right)}{S_nI(x)}\geq\frac{-\log\left(\mu\left(C_{n_k}(x)\right)\right)}{n_k}\frac{n_k}{n_k+m_k}.$$

Finally, if $n_k+m_k\leq n<n_{k+1}$ then $C_n(x)=DB$, for some $D\in\mathscr{D}_k$ and for some cylinder set B of length at least m_k such that B contains some cylinder set $C\in\mathscr{C}_{k+1}$. We then have by construction that $\mu(DC)\leq\mu(D)\,\mu_{\theta(k+1)}(C)$. Using this, it follows that for each $\varepsilon>0$ and n sufficiently large,

$$\begin{aligned}
\frac{-\log\left(\mu\left(C_n(x)\right)\right)}{S_nI(x)} &\geq \frac{-\log\left(\mu\left(C_{n_k}(x)\right)\right)-\log\mu_{\theta(k+1)}\left(C_{|B|}\left(G^{n_k}(x)\right)\right)}{S_nI(x)}\\
&\geq \frac{\left(\dim_H(\mu_1)-\varepsilon\right)S_{n_k}I(x)+\left(\dim_H(\mu_1)-\varepsilon\right)S_{|B|}I\left(G^{n_k}(x)\right)}{S_nI(x)}\\
&= \dim_H(\mu_1)-\varepsilon.
\end{aligned}$$

By combining the two latter inequalities, the assertion in (B) follows. $\square$

Remark 6.5. Note that the proof of Proposition 6.4 (ii) was inspired by the argument in [2, Theorem 6.7 (3)]. However, the considerations in [2] are restricted to expanding dynamical systems, whereas the dynamical system in Proposition 6.4 is expansive. Hence, the proof of Proposition 6.4 (ii) can be considered as giving a partial extension of the result in [2].

The following theorem gives the main result of this paper.

Theorem 6.6. *For the Hausdorff dimensions of Λ_∞ and $\Lambda_\sim$ we have*

$$\dim_H(\Lambda_\sim) = \dim_H(\Lambda_\infty) = \dim_H\left(\mathscr{L}(h_{\text{top}})\right).$$

Remark 6.7. By combining Theorem 6.6, Proposition 4.1 and Remark 4.3, and using the fact that $h_{\text{top}} < \chi_{\nu_F} \approx 0.792$, one immediately finds that the actual value of $\dim_H\left(\mathscr{L}(h_{\text{top}})\right)$ is trapped between 1 and the Hausdorff dimension of the measure of maximal entropy of the Farey map (cf. Figure 1.1). That is, we have

$$0.875 \approx \dim_H(\mathscr{L}(\chi_{\nu_F})) = \dim_H(\nu_F) < \dim_H\left(\mathscr{L}(h_{\text{top}})\right) < \dim_H(\mathscr{L}(0)) = 1.$$

Proof. For the proof of the second equality in the theorem, it is sufficient to show that

$$\mathscr{L}(h_{\text{top}} + \kappa) \subset \Lambda_\infty \subset \mathscr{L}_*(h_{\text{top}}), \text{ for each } \kappa > 0. \tag{6.4}$$

The first inclusion is just the first statement in Proposition 6.1. For the second inclusion in (6.4), let $x \in \Lambda_\infty$ be given. We then have $\lim_{n\to\infty} 2^n \lambda(T_n(x)) = 0$, which gives that for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $2^n \lambda(T_n(x)) < \varepsilon$, for all $n \geq N_\varepsilon$. Now note that we have the following chain of implications.

$$\begin{aligned} 2^n \lambda(T_n(x)) < \varepsilon \quad &\Longrightarrow \quad \lambda(T_n(x)) < \varepsilon 2^{-n} \Longrightarrow \log\lambda(T_n(x)) < -nh_{\text{top}} + \log\varepsilon \\ &\Longrightarrow \quad \ell_n(x) > h_{\text{top}} - \log\varepsilon/n. \end{aligned}$$

It follows that $\liminf_{n\to\infty} \frac{S_n I(x)}{S_n N(x)} \geq \liminf_{n\to\infty} \ell_n(x) \geq h_{\text{top}}$. This shows that $x \in \mathscr{L}_*(h_{\text{top}})$, and hence, $\Lambda_\infty \subset \mathscr{L}_*(h_{\text{top}})$. This finishes the proof of the second inclusion in (6.4), and hence finishes the proof of the second equality stated in the theorem.
For the remaining assertions of the theorem, first note that by Lemma 6.4 we have $\dim_H(\mathscr{L}(h_{\text{top}})) = \dim_H(\mathscr{L}^*(h_{\text{top}}))$. Hence, for the upper bound, it is sufficient to show that $\Lambda_\sim \subset \mathscr{L}^*(h_{\text{top}})$. In order to prove this, note that we have that $\Lambda_\sim \subset \mathscr{U} \setminus \Lambda_0$. By the second part of Proposition 5.3 we have

$$x \in \mathscr{U} \setminus \Lambda_0 \implies \limsup_{n\to\infty} \frac{a_n q_n q_{n-1}}{2^{\sum_{j=1}^n a_j}} > 0 \implies \limsup_{n\to\infty} \frac{S_n I(x)}{S_n N(x)} \geq h_{\text{top}},$$

and hence $x \in \mathscr{L}^*(h_{\text{top}})$. This finishes the proof of the upper bound $\dim_H(\Lambda_\sim) \leq \dim_H\left(\mathscr{L}(h_{\text{top}})\right)$.
For the lower bound, note that by Corollary 6.2 we have that

$$\left\{x \in \mathscr{U} : \liminf_{n\to\infty} \frac{S_n I(x)}{S_n N(x)} < h_{\text{top}} < \limsup_{n\to\infty} \frac{S_n I(x)}{S_n N(x)}\right\} \subset \Lambda_\sim.$$

Hence, it is sufficient to show that $\dim_H(\mathscr{L}(s_0,s_1)) \geq \dim_H(\mathscr{L}(s_1))$, for each $s_0 \in (0, h_{\text{top}})$ and $s_1 \in (h_{\text{top}}, \infty)$. Since the latter is an immediate consequence of Proposition 6.4, the proof of the theorem is complete. □

Let us finish the paper with the following immediate consequence of Theorem 6.6.

Corollary 6.8. *For the Hausdorff dimension of $\mathscr{U} \setminus \Lambda_0$ we have*

$$\dim_H(\nu_F) < \dim_H(\mathscr{U} \setminus \Lambda_0) = \dim_H(\mathscr{L}(h_{\text{top}})) < 1.$$

In particular, this implies the aforementioned result of Salem [30], *namely that Q is a singular function in the sense that*

$$\lambda(\Lambda_0) = 1.$$

References

[1] P. Bak, R. Bruinsma. One-dimensional Ising model and the complete devil's staircase. *Phys. Rev. Lett.*, 49(4):249–252, 1982.

[2] L. Barreira, J. Schmeling. Sets of "non-typical" points have full topological entropy and full Hausdorff dimension. *Israel J. Math.*, 116:29–70, 2000.

[3] A. Brocot. Calcul des rouages par approximation, nouvelle méthode. *Revue Chronométrique*, 6:186–194, 1860.

[4] R. Darst. The Hausdorff dimension of the non–differentiability set of the Cantor function is $[\ln 2/\ln 3]^2$. *Proc. Amer. Math. Soc.*, 119:105–108, 1993.

[5] A. Denjoy. Sur quelques points de la théorie des fonctions. *C. R. Acad. Sci. Paris*, 194:44–46, 1932.

[6] A. Denjoy. Sur une fonction réelle de Minkowski. *J. Math. Pures Appl.*, 17:105–151, 1938.

[7] K.J. Falconer. *Fractal Geometry*. Wiley, New York, 1990.

[8] K.J. Falconer. One-sided multifractal analysis and points of non-differentiability of devil's staircases. *Math. Proc. Camb. Phil. Soc.*, 136:167–174, 2004.

[9] U. Frisch, G. Parisi. On the singularity structure of fully developed turbulence. In *Turbulence and predictability in geophysical fluid dynamics and climate dynamics*, pages 84–88. North Holland Amsterdam, 1985.

[10] R.E. Gilman. A class of functions continuous but not absolutely continuous. *Ann. Math.*, 33:433–442, 1932.

[11] M. Gutzwiller. *Chaos in classical and quantum mechanics*. Interdis. Appl. Math., Springer Verlag, 1990.

[12] M. Gutzwiller, B.B. Mandelbrot. Invariant multifractal measures in chaotic Hamiltonian systems, and related structures. *Phys. Rev. Lett.*, 60:673–676, 1988.

[13] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. J. Shraiman. Fractal measures and their singularities: The characterization of strange sets. *Phys. Rev. A*, 85(33):1141–1151, 1986.

[14] G. H. Hardy, E. M. Wright. *The theory of numbers*. 4th ed., Oxford Univ. Press, 1960.

[15] S. Ito. Algorithms with mediant convergents and their metrical theory. *Osaka J. Math.*, 26 (3):557–578, 1989.

[16] V. Jarník. Zur metrischen Theorie der Diophantischen Approximationen. *Prace Math.*, 36, 2. Heft, 1928.

[17] M. Kesseböhmer, B.O. Stratmann. A multifractal formalism for growth rates and applications to geometrically finite Kleinian groups. *Ergodic Theory & Dynamical Systems*, 24 (01):141–170, 2004.

[18] M. Kesseböhmer, B.O. Stratmann. Stern-Brocot pressure and multifractal spectra in ergodic theory of numbers. *Stochastics and Dynamics*, 4 (1):77 - 84, 2004.

[19] M. Kesseböhmer, B.O. Stratmann. A multifractal analysis for Stern-Brocot intervals, continued fractions and Diophantine growth rates. *J. reine angew. Math.*, 605, 2007.

[20] A.Ya. Khintchine. *Continued fractions.* Univ. of Chicago Press, Chicago,IL., 1964.

[21] J.R. Kinney. Note on a singular function of Minkowski. *Proc. Amer. Math. Soc.*, 11:788–794, 1960.

[22] J.C. Lagarias. Number theory and dynamical systems. *Proc. of Symp. in Appl. Math.*, 46:35–72, 1992.

[23] B. Mandelbrot. *Fractals: form, chance, and dimension*. Freeman, San Francisco, 1977.

[24] P. Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*. Cambridge University Press, 1995.

[25] H. Minkowski. Zur Geometrie der Zahlen. *Gesammelte Abhandlungen*, Vol. 2, 1911; reprinted by Chelsea, New York, 43–52, 1967.

[26] J. Paradís, P. Viader. The derivative of Minkowski's $?(x)$ function. *J. Math. Annal. and Appl.*, 253:107–125, 2001.

[27] I. Richards. Continued fractions without tears. *Math. Mag.*, 54:163–171, 1981.

[28] F. Ryde. Arithmetical continued fractions. *Lunds universitets arsskrift, N. F. Adv. 2*, 22 (2):01–182, 1926.

[29] F. Ryde. On the relation between two Minkowski functions. *J. Number Theory*, 17:47–51, 1983.

[30] R. Salem. On some singular monotonic functions which are strictly increasing. *Trans. Amer. Math. Soc.*, 53:427–439, 1943.

[31] M.A. Stern. Über eine zahlentheoretische Funktion. *J. reine angew. Math.*, 55:193–220, 1858.

[32] B.O. Stratmann. Weak singularity spectra of the Patterson measure for geometrically finite Kleinian groups with parabolic elements. *Michigan Math. Jour.*, 46:573–587, 1999.

[33] R.F. Tichy, J. Uitz. An extension of Minkowski's singular function. *Appl. Math. Lett.*, 8:39–46, 1995.

[34] P. Viader, J. Paradís, L. Bibiloni. A new light on Minkowski's $?(x)$ function. *J. Number Theory*, 73:212–227, 1998.

FACHBEREICH 3 - MATHEMATIK UND INFORMATIK, UNIVERSITÄT BREMEN, D–28359 BREMEN, GERMANY

E-mail address: `mhk@math.uni-bremen.de`

MATHEMATICAL INSTITUTE, UNIVERSITY OF ST ANDREWS, ST ANDREWS KY16 9SS, SCOTLAND

E-mail address: `bos@maths.st-and.ac.uk`